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UNIT GROUPS AND CLASS NUMBERS OF REAL CYCLIC OCTIC FIELDS

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ABSTRACT. The generating polynomials of D. Shanks' simplest quadratic and cubic fields and M.-N. Gras' simplest quartic and sextic fields can be obtained by working in the group $\mathbf{PGL}_2(\mathbf{Q})$. Following this procedure and working in the group $\mathbf{PGL}_2(\mathbf{Q}(\sqrt{2}))$, we obtain a family of octic polynomials and hence a family of real cyclic octic fields. We find a system of independent units which is close to being a system of fundamental units in the sense that the index has a uniform upper bound. To do this, we use a group theoretic argument along with a method similar to one used by T. W. Cusick to find a lower bound for the regulator and hence an upper bound for the index. Via Brauer-Siegel's theorem, we can estimate how large the class numbers of our octic fields are. After working out the first three examples in §5, we make a conjecture that the index is 8. We succeed in getting a system of fundamental units for the quartic subfield. For the octic field we obtain a set of units which we conjecture to be fundamental. Finally, there is a very natural way to generalize the octic polynomials to get a family of real 2^n-tic number fields. However, to select a subfamily so that the fields become Galois over $\mathbf{Q}$ is not easy and still a lot of work on these remains to be done.

1. INTRODUCTION

To calculate the class number of an abelian algebraic number field via the analytic class number formula usually involves a calculation of the regulator of the field, which in turn requires finding a system of fundamental units. To do this, in general, is a very hard job. For real quadratic fields, there is an easy algorithm to find a fundamental unit. For higher degree real algebraic number fields, the situation is more difficult. However, for the so-called "simplest fields", we can find a system of fundamental units easily.

Daniel Shanks [8] selected a family of real cubic fields, which are generated by a root w_3 of

$$X^3 - aX^2 - (a+3)X - 1, \qquad a \in \mathbf{Z},$$

and a conjugate of w_3 is $(-w_3 - 1)/w_3$.

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In the same paper, he studied the analogous real quadratic fields. We may call them "the simplest quadratic fields", which are generated by a root w_2 of

$$X^2 - aX - 1, \qquad a \in \mathbf{Z},$$

and a conjugate of w_2 is $-1/w_2$.

Marie-Nicole Gras [3] gave analogues for cyclic quartic and sextic fields. We may call them "the simplest quartic fields", and "simplest sextic fields". The quartic fields are generated by a root w_4 of

$$X^4 - aX^3 - 6X^2 + aX + 1, \qquad a \in \mathbf{Z} - \{0, \pm 3\},$$

and a conjugate of w_4 is $(w_4 - 1)/(w_4 + 1)$. The sextic fields are generated by a root w_6 of

$$X^6 - aX^5 - (\tfrac{5}{2}a + 15)X^4 - 20X^3 + \tfrac{5}{2}aX^2 + (a+6)X + 1, \qquad a \in 2\mathbf{Z} - \{0, -6, 10, -16\},$$

and a conjugate of w_6 is $(w_6 - 1)/(w_6 + 2)$.

The main interest in these fields is that they have an explicit system of fundamental units which are relatively small and hence a relatively large class number. Gary Cornell and Larry Washington [1] pointed out a systematic way to construct these generating polynomials, which is as follows:

The quadratic ones come from the matrix

$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is of order 2 in the group $\mathbf{PGL}_2(\mathbf{Q})$, and is similar to the diagonal matrix

$$\begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}.$$

The cubic ones come from the matrix

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

which is of order 3 in the group $\mathbf{PGL}_2(\mathbf{Q})$, and is similar to the diagonal matrix

$$\begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}.$$

The quartic ones come from the matrix

$$A_4 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which is of order 4 in the group $\mathbf{PGL}_2(\mathbf{Q})$, and is similar to the diagonal matrix

$$\begin{pmatrix} 1+\zeta_4 & 0 \\ 0 & 1+\zeta_4^{-1} \end{pmatrix}.$$

The sextic ones come from the matrix

$$A_6 = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix},$$

which is of order 6 in the group $\mathbf{PGL}_2(\mathbf{Q})$, and is similar to the diagonal matrix

$$\begin{pmatrix} 1+\zeta_6 & 0 \\ 0 & 1+\zeta_6^{-1} \end{pmatrix}.$$

The construction for each case goes like this:

Given θ. Let $\theta_i = A^{i-1}\theta$, $i = 1, 2, \dots, n$, where A acts on θ by

$$A\theta = \frac{\alpha\theta+\beta}{\gamma\theta+\delta} \quad \text{if } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then θ satisfies the polynomial

$$X^n - a_1X^{n-1} + a_2X^{n-2} - a_3X^{n-3} + \cdots,$$

where $a_1 = \sum_{i=1}^n \theta_i$, $a_2 = \sum_{i<j} \theta_i\theta_j$, etc.

Using this method, we are trying to find "the simplest octic fields". So we need a matrix A_8, which is of order 8 in the group $\mathbf{PGL}_2(\mathbf{Q})$. Since $\zeta_8 = \sqrt{2}/2 + i(\sqrt{2}/2)$, we cannot find a matrix of order 8 in the group $\mathbf{PGL}_2(\mathbf{Q})$. In fact, if A is of order n in $\mathbf{PGL}_2(\mathbf{Q})$, then A is conjugate in $\mathbf{GL}_2(\mathbf{C})$ to the matrix

$$\lambda \begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for some } \lambda.$$

Therefore

$$\frac{(\operatorname{tr} A)^2}{\det A} = \frac{(\zeta_n+1)^2}{\zeta_n} = 2 + 2\cos\frac{2\pi}{n}.$$

This is in $\mathbf{Q}$ only for $n = 1, 2, 3, 4, 6$. Thus we have to work on the quadratic field $\mathbf{Q}(\sqrt{2})$.

So now, we are looking for a matrix A_8, which is order 8 in the group $\mathbf{PGL}_2(\mathbf{Q}(\sqrt{2}))$. Of course, we start with the matrices

$$\begin{pmatrix} 1+\zeta_8 & 0 \\ 0 & 1+\zeta_8^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \zeta_{16} & 0 \\ 0 & \zeta_{16}^{-1} \end{pmatrix}.$$

However, it turns out that the diagonal matrix

$$D_8 = \begin{pmatrix} \sqrt{2}(1+\zeta_8) & 0 \\ 0 & \sqrt{2}(1+\zeta_8^{-1}) \end{pmatrix}$$

is a better one to start with. Note that D_8 has trace 2ε and norm $2\varepsilon+2$, where $\varepsilon = \sqrt{2}+1$. Any matrix which is similar to D_8 must have the same trace and norm. One such A is $\left(\begin{smallmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{smallmatrix}\right)$, since $\varepsilon^2 = 2\varepsilon + 1$.

Using this matrix, we will construct a family of octic polynomials in §2 and select an interesting subfamily so that the generating fields are totally real cyclic octic fields. In §3, we will study these fields, finding a system of independent units, which is close to being a system of fundamental units. To show how close it is, we get a lower bound on the regulator and hence an upper bound on the index in §4. In fact, we obtain an upper bound which works for all fields in the family. Also via Brauer-Siegel's theorem we can estimate how large the class number is. After that we will do some calculations in §5 on the first few examples and then make a conjecture that the index is 8. From this conjecture, we succeed in getting a system of fundamental units for the quartic subfield. For the octic field, we obtain a set of units which we conjugate to be fundamental. Finally, there is a very natural way to generalize the polynomials we constructed in §2 to get a family of real 2^n-tic algebraic number fields. However, to select a subfamily so that the fields are Galois over $\mathbf{Q}$ is not easy and still a lot of work on these remains to be done.

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2. The construction of the octic polynomials

Let $\varepsilon = \sqrt{2}+1$ and let $A = \begin{pmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{pmatrix}$. Then A is of order 8 in the group $\mathbf{PGL}_2(\mathbf{Q}(\sqrt{2}))$, since

$$A^2 \sim \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A^3 \sim \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad A^4 \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^5 \sim \begin{pmatrix} -1 & -\varepsilon \\ \varepsilon & -1 \end{pmatrix},$$
$$A^6 \sim \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A^7 \sim \begin{pmatrix} -\varepsilon & -1 \\ 1 & -\varepsilon \end{pmatrix}, \quad A^8 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\theta_1 = \theta$ and let $\theta_i = A^{i-1}\theta$, $i = 2, \dots, 8$. Writing them out, we have

$$\begin{gathered} \theta_1 = \theta, \qquad \theta_2 = \frac{\varepsilon\theta - 1}{\theta + \varepsilon}, \qquad \theta_3 = \frac{\theta - 1}{\theta + 1}, \\ \theta_4 = \frac{\theta - \varepsilon}{\varepsilon\theta + 1}, \qquad \theta_5 = \frac{-1}{\theta}, \qquad \theta_6 = \frac{-\theta - \varepsilon}{\varepsilon\theta - 1}, \\ \theta_7 = \frac{-\theta - 1}{\theta - 1}, \qquad \theta_8 = \frac{-\varepsilon\theta - 1}{\theta - \varepsilon}. \end{gathered} \tag{2.1}$$

Clearly, $\prod_{i=1}^{8} \theta_i = 1$. Therefore θ satisfies the octic polynomial

$$X^8 - a_1X^7 + a_2X^6 - a_3X^5 + a_4X^4 - a_5X^3 + a_6X^2 - a_7X + 1,$$

where $a_1 = \sum_{i=1}^{8} \theta_i$, $a_2 = \sum_{i<j} \theta_i\theta_j$, etc. A straightforward calculation yields that

$$a_1 = \theta + \frac{-1}{\theta} + \frac{-2}{\theta - 1} + \frac{-2}{\theta + 1} + \frac{-2 - 2\varepsilon}{\theta - \varepsilon} + \frac{-2 - 2\varepsilon}{\theta + \varepsilon} + \frac{-6 + 2\varepsilon}{\theta - \varepsilon^{-1}} + \frac{-6 + 2\varepsilon}{\theta + \varepsilon^{-1}},$$
$$a_2 = -28, \quad a_3 = -7a_1, \quad a_4 = 70, \quad a_5 = 7a_1, \quad a_6 = -28, \quad a_7 = -a_1.$$

So we have a family of octic polynomials

$$P(X) = X^8 - a_1X^7 - 28X^6 + 7a_1X^5 + 70X^4 - 7a_1X^3 - 28X^2 + a_1X + 1. \tag{2.2}$$

Remark. It takes a lot of time to get the above straightforward calculation done. Don Zagier pointed out that we have a much quicker method to avoid such a lengthy calculation. For our case, it goes like this: The matrix A is similar to the diagonal matrix

$$D = \begin{pmatrix} \varepsilon + i & 0 \\ 0 & \varepsilon - i \end{pmatrix}.$$

Therefore $A = B^{-1}DB$ for some invertible B. A calculation shows that we may choose B to be the matrix $\left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$. Thus θ satisfies the following polynomial:

$$P(X) = \prod_{\nu=1}^{8} (X - A^{\nu}\theta).$$

Hence we have

$$\begin{aligned} P(X) = 0 \Leftrightarrow X = A^k\theta &= B^{-1}D^kB\theta, \\ \Leftrightarrow BX &= \left(\frac{\varepsilon + i}{\varepsilon - i}\right)^k B\theta, \\ \Leftrightarrow (BX)^8 &= (B\theta)^8, \quad \text{since } \frac{\varepsilon + i}{\varepsilon - i} = \zeta_8, \\ \Leftrightarrow \left(\frac{X+i}{X-i}\right)^8 &= \left(\frac{\theta + i}{\theta - i}\right)^8. \end{aligned}$$

Therefore

$$\begin{aligned} P(X) &= \frac{(\theta - i)^8(X+i)^8 - (\theta + i)^8(X - i)^8}{(\theta - i)^8 - (\theta + i)^8} \\ &= \sum_{\nu=0}^{8} \binom{8}{\nu} X^{8-\nu}\alpha_\nu, \end{aligned}$$

where

$$\alpha_\nu = \frac{i^\nu(\theta - i)^8 - (-i)^\nu(\theta + i)^8}{(\theta - i)^8 - (\theta + i)^8}.$$

It is easy to see that

$$\alpha_\nu = \begin{cases} (-1)^{\nu/2} & \text{if } \nu \text{ is even,} \\ (-1)^{(\nu-1)/2}\alpha_1 & \text{if } \nu \text{ is odd.} \end{cases}$$

Let $a_1 = 8\alpha_1$; then we get the polynomial $P(X)$ in equation (2.2). Note that $\alpha_1 \in \mathbf{R}$, since $\overline{\alpha}_1 = \alpha_1$.

We may write the polynomial $P(X)$ in equation (2.2) as

$$X^8 - 28X^6 + 70X^4 - 28X^2 + 1 - a_1X(X^2 - 1)(X^2 - \varepsilon^2)(X^2 - \varepsilon^{-2}). \tag{2.3}$$

An easy calculation via equation (2.3) yields

$$\begin{array}{lll} P(-\infty) > 0, & P(-\varepsilon) < 0, & P(-1) = 16 > 0, \\ P(-\varepsilon^{-1}) < 0, & P(0) = 1 > 0, & P(\varepsilon^{-1}) < 0, \\ P(1) = 16 > 0, & P(\varepsilon) < 0, & P(+\infty) > 0. \end{array}$$

Hence $P(X) = 0$ has eight distinct real roots. Let θ be the largest one. Then we have the following results:

$$\begin{gathered}\theta_1 \in (\varepsilon, \infty), \qquad \theta_2 \in (1, \varepsilon), \qquad \theta_3 \in (\varepsilon^{-1}, 1), \\ \theta_4 \in (0, \varepsilon^{-1}), \qquad \theta_5 \in (-\varepsilon^{-1}, 0), \qquad \theta_6 \in (-1, -\varepsilon^{-1}), \\ \theta_7 \in (-\varepsilon, -1), \qquad \theta_8 \in (-\infty, -\varepsilon).\end{gathered} \tag{2.4}$$

They are all units in the ring of algebraic integers of the field $\mathbf{Q}(\theta, \sqrt{2})$ if a_1 is an algebraic integer of the field $\mathbf{Q}(\sqrt{2})$. Now let us check the irreducibility of the polynomial $P(X)$. For this, we have the following proposition. The proof is very long, but straightforward (see Shen [9]).

Proposition 1. *The octic polynomial $P(X)$ in equation* (2.2) *is irreducible over the field $\mathbf{Q}$ of rational numbers for $a_1 \in \mathbf{Z} - \{0, \pm 6, \pm 15\}$.*

Remark. For $a_1 = 0, \pm 6, \pm 15$, the polynomial $P(X)$ does factor, and factors in the following way:

$$\begin{gathered}(X^4 + 4X^3 - 6X^2 - 4X + 1)(X^4 - 4X^3 - 6X^2 + 4X + 1), \\ (X^4 \pm 2X^3 - 6X^2 \mp 2X + 1)(X^4 \mp 8X^3 - 6X^2 \pm 8X + 1), \\ (X^4 \pm X^3 - 6X^2 \mp X + 1)(X^4 \mp 16X^3 - 6X^2 \pm 16X + 1).\end{gathered}$$

Now let θ be a root of $P(X) = 0$ for $a_1 \in \mathbf{Z} - \{0, \pm 6, \pm 15\}$, and let $\theta_i = A^{i-1}\theta$, $i = 1, 2, 3, \dots, 8$. Then they are all roots of $P(X) = 0$. Let $y = \frac{1}{2}(\theta_1 + \theta_5)$ and $z = \frac{1}{4}(\theta_1 + \theta_3 + \theta_5 + \theta_7)$. Then we have the following proposition. For a proof, see Shen [9].

Proposition 2. (a) *The minimal polynomial of y over $\mathbf{Q}$ is*

$$X^4 - \tfrac{1}{2}a_1 X^3 - 6X^2 + \tfrac{1}{2}a_1 X + 1,$$

and hence $\mathbf{Q}(y)$ is a "simplest quartic field".

(b) *The minimal polynomial of z over $\mathbf{Q}$ is $X^2 - \frac{1}{4}a_1 X - 1$, and hence $\mathbf{Q}(z) = \mathbf{Q}(\sqrt{a_1^2 + 64})$.*

(c) *We have $\mathbf{Q}(z) = \mathbf{Q}(\sqrt{2})$ if $a_1 \in \{a_1 \in \mathbf{Z} | a_1 + b_1\sqrt{2} = \pm 8\varepsilon^{2n+1},\ n \in \mathbf{Z}\}$.*

Remark. In (b), when $a_1^2 + 64$ is a square in $\mathbf{Z}$, then $a_1 = 0, \pm 6, \pm 15$. These are the degenerate cases, which made the polynomial $P(X)$ reducible in Proposition 1.

Because of the special form for a_1 in the last statement of Proposition 2, put $a_1 = 8a$, and let us work with the octic polynomial of the following form from now on:

$$Q(X) = X^8 - 8aX^7 - 28X^6 + 56aX^5 + 70X^4 - 56aX^3 - 28X^2 + 8aX + 1 \tag{2.5}$$

for $a \in \mathbf{Z}$. Let ρ be the largest root of $Q(X) = 0$ and let $\rho_i = \sigma^{i-1}\rho$ for $i = 1, 2, 3, \dots 8$, where $\sigma \in \mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$ is determined by

$$\rho \mapsto \frac{\varepsilon\rho - 1}{\rho + \varepsilon}.$$

Note that $\rho_i = \theta_i$ for $i = 1, 2, 5, 6$. However,

$$\rho_3 = \theta_7, \qquad \rho_4 = \theta_8, \qquad \rho_7 = \theta_3, \qquad \rho_8 = \theta_4.$$

We have a different ordering since $\sigma(\varepsilon) = -\varepsilon^{-1}$.

Let $K_a = \mathbf{Q}(\rho)$. Since $K_{-a} = K_a$ for $a \in \mathbf{Z}$, we may assume $a > 0$, $a \in \mathbf{Z}$ hereafter. As above, we let $y = \frac{1}{2}(\rho_1 + \rho_5)$ and $z = \frac{1}{4}(\rho_1 + \rho_3 + \rho_5 + \rho_7)$. Then $y = \frac{1}{2}(\rho - 1/\rho)$ and $z = \frac{1}{2}(y - 1/y)$, and hence we have the following proposition.

Proposition 3. (a) *The polynomial* $Q(X)$ *in equation* (2.5) *is irreducible over* $\mathbf{Q}$.

(b) *The minimal polynomial of* y *over* $\mathbf{Q}$ *is* $X^4 - 4aX^3 - 6X^2 + 4aX + 1$.

(c) *The minimal polynomial of* z *over* $\mathbf{Q}$ *is* $X^2 - 2aX - 1$.

(d) *The minimal polynomial of* ρ *over* $\mathbf{Q}(y)$ *is* $X^2 - 2yX - 1$.

(e) *The minimal polynomial of* y *over* $\mathbf{Q}(z)$ *is* $X^2 - 2zX - 1$.

(f) *The minimal polynomial of* ρ *over* $\mathbf{Q}(z)$ *is* $X^4 - 4zX^3 - 6X^2 + 4zX + 1$.

Remarks. (a) We have

$$Q(X) = \Re(X + i)^8 - a\Im(X + i)^8,$$

$$X^4 - 4aX^3 - 6X^2 + 4aX + 1 = \Re(X + i)^4 - a\Im(X + i)^4,$$

and

$$X^2 - 2aX - 1 = \Re(x + i)^2 - a\Im(x + i)^2.$$

(b) We may say that $\mathbf{Q}(\rho)$ is a simplest quartic field over $\mathbf{Q}(z)$.

Let $y_i = \sigma^{i-1} y$, $i = 1, 2, 3, 4$, and let $z_i = \sigma^{i-1} z$, $i = 1, 2$. Proposition 3 tells us that

$$z_1 = a + \sqrt{a^2 + 1}, \qquad z_2 = a - \sqrt{a^2 + 1}, \tag{2.6}$$

$$y_i = z_i + \sqrt{z_i^2 + 1}, \quad y_{i+2} = z_i - \sqrt{z_i^2 + 1}, \qquad i = 1, 2, \tag{2.7}$$

$$\theta_i = y_i + \sqrt{y_i^2 + 1}, \quad \theta_{i+4} = y_i - \sqrt{y_i^2 + 1}, \qquad i = 1, 2, 3, 4. \tag{2.8}$$

Thus we have

$$z_1 z_2 = -1; \qquad y_i y_{i+2} = -1, \quad i = 1, 2; \qquad \theta_i \theta_{i+4} = -1, \quad i = 1, 2, 3, 4.$$

Note that if $\rho = \theta$ then

$$\begin{aligned} &\rho_1 = \theta_1, \quad \rho_2 = \theta_2, \quad \rho_3 = \theta_7, \quad \rho_4 = \theta_8, \\ &\rho_5 = \theta_5, \quad \rho_6 = \theta_6, \quad \rho_7 = \theta_3, \quad \rho_8 = \theta_4. \end{aligned}$$

Also, as $a \to \infty$, the above formulas imply that

$$z_1 \sim 2a, \qquad z_2 \to 0, \tag{2.9}$$

$$y_1 \sim 4a, \qquad y_2 \to 1, \qquad y_3 \to 0, \qquad y_4 \to -1, \tag{2.10}$$

$$\rho_1 \sim 8a, \qquad \rho_2 \to \varepsilon, \qquad \rho_3 \to -1, \qquad \rho_4 \to -\varepsilon. \tag{2.11}$$

For later use we express each y_i in term of a as follows:

$$
\begin{aligned}
y_1 &= a + \sqrt{a^2+1} + \sqrt{2(a^2+1) + 2a\sqrt{a^2+1}}\,, \\
y_2 &= a - \sqrt{a^2+1} + \sqrt{2(a^2+1) - 2a\sqrt{a^2+1}}\,, \\
y_3 &= a + \sqrt{a^2+1} - \sqrt{2(a^2+1) + 2a\sqrt{a^2+1}}\,, \\
y_4 &= a - \sqrt{a^2+1} - \sqrt{2(a^2+1) - 2a\sqrt{a^2+1}}\,.
\end{aligned} \tag{2.12}
$$

3. The totally real cyclic octic fields

Let $S = \{a_n \in \mathbf{Z} | a_n + b_n\sqrt{2} = \varepsilon^{2n+1},\ n \in \mathbf{N}\}$, and let $K_n = K_{a_n}$, where $a_n \in S$. Then $K_n/\mathbf{Q}$ is Galois (since $\sqrt{2} \in K_n$) and σ is an automorphism of $K_n/\mathbf{Q}$. We have the following theorem from the previous proposition.

Theorem 1. (a) *For $a_n \in S$, the field K_n is a totally real cyclic octic field, whose Galois group* $\mathrm{Gal}(K_n/\mathbf{Q})$ *is* $\langle\sigma\rangle \simeq \mathbf{Z}/8\mathbf{Z}$.

(b) *The unique quartic subfield of K_n is a "simplest quartic field" $\mathbf{Q}(y_n)$, where y_n is a root of the polynomial*

$$X^4 - 4a_nX^3 - 6X^2 + 4a_nX + 1\,.$$

(c) *The unique quadratic subfield of K_n is always* $\mathbf{Q}(\sqrt{2})$.

Remark. For $a_n \in S$, $a_n^2 + 1 = 2b_n^2$ and $b_n > 0$. Thus $\sqrt{a_n^2+1} = b_n\sqrt{2}$. Formulas in (2.12) and the above relation give us that

$$y_n = a_n + b_n\sqrt{2} + \sqrt{2\sqrt{2}b_n(a_n + b_n\sqrt{2})}\,.$$

But $a_n + b_n\sqrt{2} = \varepsilon^{2n+1}$, $n \in \mathbf{N}$. So we have

$$y_n = \varepsilon^{2n+1} + \varepsilon^n\sqrt{b_n(4+2\sqrt{2})}\,.$$

Claim. The quotient

$$\frac{b_m(4+2\sqrt{2})}{b_n(4+2\sqrt{2})} = \frac{b_m}{b_n}$$

is not a square in $\mathbf{Q}(\sqrt{2})$ for infinitely many m and n. Therefore there are infinitely many distinct octic fields K_n.

Proof. We now prove this claim. For any b_m, $a_m^2 + 1 = 2b_m^2$. Say $b_m = s^2b_n$ for some $s \in \mathbf{Z}[\sqrt{2}]$. Then $b_m^2 = s^4b_n^2$, and hence $a_m^2 + 1 = 2b_n^2s^4$. A result due to Siegel [10] implies that $a^2 + 1 = cs^4$ has only finite many integral solutions. Thus it follows easily that for any b_m, there are only finitely many b_n which differ from b_m be a square in $\mathbf{Z}[\sqrt{2}]$. Hence the claim is true.

We shall eventually want b_n squarefree in $\mathbf{Z}$. It is probably very hard to prove this happens infinitely often, though it is surely true. Since $a_n - b_n\sqrt{2} = -\varepsilon^{-(2n+1)}$,

$$b_n = \frac{\sqrt{2}}{4}(\varepsilon^{2n+1} + \varepsilon^{-(2n+1)}),$$

and hence

$$b_n = \frac{\sqrt{2}}{4}\varepsilon^{-(2n+1)}(\varepsilon^{4n+2} + 1).$$

The situation is similar to proving that there are infinitely many squarefree Mersenne numbers $2^n - 1$, which is not yet proved.

Our main interest in K_n is that we can give a system of independent units explicitly, hence an upper bound on the regulator which is relatively small. We know that $\rho_1, \rho_2, \rho_3, \rho_4, y_1, y_2$ and ε are all units in the ring of algebraic integers of the real cyclic octic field $K_n = \mathbf{Q}(\rho)$. Even better, we have the following theorem.

Theorem 2. *The units $\rho_1, \rho_2, \rho_3, \rho_4, y_1, y_2$, and ε in the ring of algebraic integers of the real cyclic octic field K_n are independent.*

Proof. See Shen [9]. Note that it also follows from the fact (see below) that $R_\rho \to \infty$ as $a \to \infty$ that the result is true for sufficiently large a.

Next, we calculate the regulator

$$R_1 = \operatorname{Reg}(\rho_1, \rho_2, \rho_3, \rho_4, y_1, y_2, \varepsilon)$$

of ρ_1, ρ_2, ρ_3, ρ_4, y_1, y_2, and ε. A calculation yields

$$R_1 = 16\log\varepsilon(\log^2|y_1| + \log^2|y_2|)R_\rho, \tag{3.1}$$

where R_ρ is the regulator of ρ_1, ρ_2, ρ_3, and ρ_4. In fact, it is

$$R_\rho = \sum_{i=1}^{4} A_i^4 + 2\sum_{i=1}^{2} A_i^2 A_{i+2}^2 + 4(A_1A_2 + A_3A_4)(A_1A_4 - A_2A_3), \tag{3.2}$$

where $A_i = \log|\rho_i| = -\log|\rho_{i+4}|$, $i = 1, 2, 3, 4$.

Remark. From (2.10) and (2.11), as $a \to \infty$ we have

$$R_\rho \sim (\log^2(8a) + 2\log^2\varepsilon)^2,$$

and hence we have the asymptotic approximation for R_1 as $a \to \infty$, as follows:

$$R_1 \sim 16\log\varepsilon\log^2(4a)(\log^2(8a) + 2\log^2\varepsilon)^2 \sim 16\log\varepsilon\log^6 a. \tag{3.3}$$

The discriminant $d(K_n)$ of $K_n = \mathbf{Q}(\rho)$ may be computed as follows: For $a \in S$,

$$a + b\sqrt{2} = (\sqrt{2} + 1)^{2n+1} \quad \text{and} \quad a^2 + 1 = 2b^2.$$

Both a and b are odd. We will make the following assumption henceforth:

Assume that b is squarefree in $\mathbf{Z}$.

Since the discriminant $d(\mathbf{Q}(\sqrt{2}))=8$, the conductor of $\mathbf{Q}(\sqrt{2})$ is 8. From the remark right after Theorem 1, we know that

$$y=(\sqrt{2}+1)^{2n+1}+(\sqrt{2}+1)^n\sqrt{2}\sqrt{(2+\sqrt{2})b},$$

and hence we have

$$\mathbf{Q}(y)=\mathbf{Q}\left(\sqrt{(2+\sqrt{2})b}\right).$$

Proposition 4. *The set* $\{1,\sqrt{(2+\sqrt{2})b}\}$ *forms an integral basis of* $\mathbf{Q}(y)$ *over* $\mathbf{Q}(\sqrt{2})$.

Proof. See Shen [9].

Proposition 5. *The discriminant* $d(\mathbf{Q}(y))$ *of the quartic subfield* $\mathbf{Q}(y)$ *of* K_n *is* $2^{11}b^2$.

Proof. By Proposition 4, we have

$$d(\mathbf{Q}(y))=d\left(\sqrt{(2+\sqrt{2})b}\right).$$

Using the tower property of discriminants, we have

$$\begin{aligned}
d\left(\sqrt{(2+\sqrt{2})b}\right)&=(d(\mathbf{Q}(\sqrt{2})))^2N_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}}\left(d_{\mathbf{Q}(y)/\mathbf{Q}(\sqrt{2})}\left(\sqrt{(2+\sqrt{2})b}\right)\right),\\
&=8^2N_{\mathbf{Q}(\sqrt{2})}(4b(2+\sqrt{2})),\\
&=2^6 4^2 b^2(2+\sqrt{2})(2-\sqrt{2}),
\end{aligned}$$

and hence $d(\mathbf{Q}(y))=2^{11}b^2$.

There are eight Dirichlet characters corresponding to the field K_n. Denote the trivial one by χ_0, the quadratic one by χ_2, the quartic ones by χ_4, $\chi_4^3=\overline{\chi}_4$, and the octic ones by χ_8, χ_8^3, $\chi_8^5=\overline{\chi}_8^3$, $\chi_8^7=\overline{\chi}_8$. Clearly $f_{\chi_0}=1$ and $f_{\chi_2}=8$. What are f_{χ_4} and f_{χ_8}? We have the following proposition.

Proposition 6. *The conductor of* χ_4 *is* $f_{\chi_4}=16b$.

Proof. From the conductor-discriminant formula, we obtain

$$2^{11}b^2=d(\mathbf{Q}(y))=f_{\chi_0}f_{\chi_2}f_{\chi_4}f_{\overline{\chi}_4}=8f_{\chi_4}^2.$$

The result follows immediately.

To calculate the conductor f_{χ_8}, we have to work a little bit harder. But, it is easy to see the possible candidates as follows: A calculation yields

$$\mathbf{Q}(\rho)=\mathbf{Q}\left(\sqrt{(2+\sqrt{2})\left(2b+(\sqrt{2}+1)^n\sqrt{(2+\sqrt{2})b}\right)}\right).$$

Therefore the discriminant of the octic field $K_n=\mathbf{Q}(\rho)$ must divide

$$(2^{11}b^2)^2N_{\mathbf{Q}(y)/\mathbf{Q}}\left(d_{\mathbf{Q}(\rho)/\mathbf{Q}(y)}\sqrt{(2+\sqrt{2})\left(2b+(\sqrt{2}+1)^n\sqrt{(2+\sqrt{2})b}\right)}\right),$$

which is $2^{33}b^6$. Again invoking the conductor-discriminant formula, we have

$$2^3(2^4b)^2f_{\chi_8}^4|2^{33}b^6.$$

Hence $f_{\chi_8} \leq 32b$. Since $f_{\chi_4} = 16b$, we conclude that f_{χ_8} is either $16b$ or $32b$.

Theorem 3. *The conductor of* χ_8 *is* $f_{\chi_8} = 32b$*, and hence* $d(K_n) = 2^{31}b^6$.

Proof. See Shen [9]. Since f_{χ_8} is either $16b$ or $32b$, a much shorter argument is as follows: We can write $\chi_8 = \psi\theta$, where the conductor f_ψ of ψ is a power of 2 and 2 does not divide the conductor of θ. Since 2 ramifies in $\mathbf{Q}(\sqrt{2})$, which corresponds to the character χ_8^4, we must have $\psi^4 \neq 1$. Therefore ψ has order 8. This can only happen if the conductor of ψ is at least 32, since otherwise $(\mathbf{Z}/\mathbf{f}_\psi\mathbf{Z})^\times$ does not have a cyclic quotient of order 8. Therefore 32 divides f_{χ_8}, which yields the result.

4. An estimate for the regulator

We want to estimate (and hopefully calculate) the class number of the real cyclic octic field $K_n = \mathbf{Q}(\rho)$. So let us look at the analytic class number formula:

$$\frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{d}} = \prod_{\chi\neq 1} L(1, \chi).$$

For K_n, $r_1 = 8$, $r_2 = 0$, $w = 2$, $d = 2^{31}b^6$, and therefore

$$\frac{hR}{2^8\sqrt{2}b^3} = L(1, \chi_2)|L(1, \chi_4)|^2|L(1, \chi_8)|^2|L(1, \chi_8^3)|^2.$$

We know that $L(1, \chi_2) = \log\varepsilon/\sqrt{2}$, since $h(\mathbf{Q}(\sqrt{2})) = 1$. So we have

$$h = \frac{2^8b^3\log\varepsilon}{R}|L(1, \chi_4)|^2|L(1, \chi_8)|^2|L(1, \chi_8^3)|^2. \tag{4.1}$$

To calculate $|L(1, \chi)|$, we may use the following (since χ is even)

$$|L(1, \chi)| = \frac{2}{\sqrt{f}}\left|\sum_{k<f/2} \chi(k)\log\sin\frac{k\pi}{f}\right|, \tag{4.2}$$

which is part of Theorem 46 in Marcus' book [7].

So now it is clear that, in an algebraic number field, estimating the class number h is equivalent to estimating the regulator R. In general, it is hard to estimate R. However, Theorem 2 provides us hope.

Recall formula (3.1),

$$R_1 = 16\log\varepsilon(\log^2|y_1| + \log^2|y_2|)R_\rho.$$

Let $q = R_1/R$; then q is an integer. In fact, q is the index $[U : U_1]$, where U is the unit group of K_n and $U_1 = \langle -1, \varepsilon, y_1, y_2, \rho_1, \rho_2, \rho_3, \rho_4\rangle$. We may estimate R_1 in terms of a. To determine q, we need a lower bound for R. One such lower bound due to J. H. Silverman [11] says that $R > c_8 \log^4(\gamma_8 D_8)$, where $D_8 = |d(K_n)| = 2^{31} b^6$, and c_8, γ_8 are constants depending on K_n. Since

$$D_8 = 2^{28}(2b^2)^3 = 2^{28}(a^2+1)^3,$$

$R > d_8 \log^4 a$, where d_8 is a constant. This lower bound is not good enough for our purpose, since $R_1 \sim 16 \log \varepsilon \log^6 a$. So we need some other technique to get a lower bound of order $\log^6 a$.

In the following, we denote the norm from K_n to the quartic field $\mathbf{Q}(y)$ by $N_{8/4}$, the norm from $\mathbf{Q}(y)$ to $\mathbf{Q}(\sqrt{2})$ by $N_{4/2}$, etc.

Let $U_0 = \{\eta \in U | N_{8/4}(\eta) = \pm 1\}$. Then $U/\{\pm 1\}$ and $U_0/\{\pm 1\}$ are both $\mathbf{Z}[G]$-modules, where

$$G = \mathrm{Gal}(K_n/\mathbf{Q}) = \langle \sigma \rangle \simeq \mathbf{Z}/8\mathbf{Z}.$$

Let $N = 1 + \sigma^4$ be the norm $N_{8/4}$. Then we have $\mathbf{Z}[G]/\langle N \rangle \simeq \mathbf{Z}[\zeta_8]$, which is a principal ideal domain. We may regard $U_0/\{\pm 1\}$ as a $\mathbf{Z}[\zeta_8]$-module. Since $\rho_1, \rho_2, \rho_3, \rho_4 \in U_0$ are independent, the $\mathbf{Z}$-rank of $U_0/\{\pm 1\}$ is at least 4. Let U_y be the unit group of $\mathbf{Q}(y)$, and let $U_{y0} = \{\xi \in U_y | N_{4/2}(\xi) = \pm 1\}$.

The same argument yields that $U_{y0}/\{\pm 1\}$ is a $\mathbf{Z}[\zeta_4]$-module, and the $\mathbf{Z}$-rank of $U_{y0}/\{\pm 1\}$ is at least 2, since $y_1, y_2 \in U_{y0}$ are independent. Clearly, the $\mathbf{Z}$-rank of the subgroup generated by ε, U_{y0}, and U_0 is at most 7. Therefore, the $\mathbf{Z}$-rank of $U_0/\{\pm 1\}$ must be 4, and that of $U_{y0}/\{\pm 1\}$ must be 2. Hence $U_0/\{\pm 1\}$ is a free $\mathbf{Z}[\zeta_8]$-module of rank 1, and $U_{y0}/\{\pm 1\}$ is a free $\mathbf{Z}[\zeta_4]$-module of rank 1. Therefore, there exist units $\eta_1 \in U_0$ and $\xi_1 \in U_{y0}$ such that $\{\eta_i = \eta_1^{\sigma^{i-1}} | i = 1, 2, 3, 4\}$ generates $U_0/\{\pm 1\}$, and $\{\xi_i = \xi_1^{\sigma^{i-1}} | i = 1, 2\}$ generates $U_{y0}/\{\pm 1\}$.

Let U' be the group generated by the elements $-1, \varepsilon, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4$, and let $R' = \mathrm{Reg}(\varepsilon, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4)$ be the regulator. Then we have

$$q = [U : U_1] = [U : U'][U' : U_1] = q_1 q_1',$$

where $q_1 = [U : U']$ and $q_1' = [U' : U_1]$.

Let $i_y = [U_y : \langle -1, \varepsilon, \xi_1, \xi_2 \rangle]$. We may calculate q_1 as follows:

Proposition 7. *For $\xi \in U_y$, we have*

$$N_{4/1}(\xi) = 1 \Leftrightarrow \xi \in \langle -1, \varepsilon, \xi_1, \xi_2 \rangle.$$

Proof. For any $\xi \in U_y$ with $N_{4/1}(\xi) = 1$, we have $N_{4/2}(\xi) = \pm \varepsilon^a$ for some $a \in \mathbf{Z}$. Thus

$$1 = N_{4/1}(\xi) = N_{2/1}(N_{4/2}(\xi)) = N_{2/1}(\pm \varepsilon^a) = (-1)^a,$$

and hence a is even, say $a = 2A$, for some $A \in \mathbf{Z}$. Therefore $N_{4/2}(\xi) = \pm \varepsilon^{2A}$, and so $N_{4/2}(\xi/\varepsilon^A) = \pm 1$. Hence we have $\xi/\varepsilon^A \in U_{y0} = \langle -1, \xi_1, \xi_2 \rangle$, which means $\xi \in \langle -1, \varepsilon, \xi_1, \xi_2 \rangle$. The converse is obvious. This completes the proof.

Corollary 1. *If* $N_{4/1}(\xi) = 1$ *for all* $\xi \in U_y$, *then* $i_y = 1$.

Corollary 2. *If* $N_{4/1}(\xi) = -1$ *for some* $\xi \in U_y$, *then* $i_y = 2$.

Proof. Let $E_k = \{\xi \in U_y | N_{4/1}(\xi) = k\}$, $k = 1, -1$. Then the above proposition just says $E_1 = \langle -1, \varepsilon, \xi_1, \xi_2 \rangle$. Choose $\xi_0 \in E_{-1}$. Then

$$E_{-1} = \xi_0 E_1, \quad U_y = E_1 \cup E_{-1}, \quad \text{and} \quad E_1 \cap E_{-1} = \varnothing.$$

The result follows.

Proposition 8. *For* $\eta \in U$, *we have*

$$N_{8/1}(\eta) = 1 \Leftrightarrow \eta^2 \in U'.$$

Proof. For any $\eta \in U$ with $N_{8/1}(\eta) = 1$, Proposition 7 implies

$$N_{8/4}(\eta) = \pm \varepsilon^a \xi_1^b \xi_2^c \quad \text{for some } a, b, c \in \mathbf{Z},$$

since $N_{4/1}(N_{8/4}(\eta)) = N_{8/1}(\eta) = 1$. Therefore $N_{8/4}(\eta^2) = \varepsilon^{2a}\xi_1^{2b}\xi_2^{2c}$, and hence $N_{8/4}(\eta^2/\varepsilon^a\xi_1^b\xi_2^c) = 1$. This gives us

$$\frac{\eta^2}{\varepsilon^a \xi_1^b \xi_2^c} \in U_0 = \langle -1, \eta_1, \eta_2, \eta_3, \eta_4 \rangle,$$

and we have $\eta^2 \in \langle -1, \varepsilon, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4 \rangle = U'$. Conversely, write $\eta^2 = \pm \varepsilon^a \xi_1^b \xi_2^c \eta_1^d \cdots \eta_4^e$. Then $N_{8/4}(\eta)^2 = \varepsilon^{2a}\xi_1^{2b}\xi_2^{2c}$, and hence $N_{8/4}(\eta) = \pm\varepsilon^a\xi_1^b\xi_2^c$. Therefore, we have

$$N_{4/2}(N_{8/4}(\eta)) = \varepsilon^{2a}(\pm 1), \quad N_{8/1}(\eta) = N_{2/1}(\pm\varepsilon^{2a}) = (-1)^{2a} = 1.$$

This completes the proof.

Let $U_0' = \{\eta \in U | N_{8/1}(\eta) = 1\}$. The same argument as in Proposition 7 yields that $[U : U_0'] = 1$ or 2. Proposition 8 says that $U_0' = \{\eta \in U | \eta^2 \in U'\}$. How large is $[U_0' : U']$? This index must be 1, 2, 4, or 8. To see this, we write

$$\eta^2 = \pm\varepsilon^a \xi_1^b \xi_2^c \eta_1^d \cdots \eta_4^e \quad \text{for } \eta \in U_0'.$$

Assume a, b, c are even, say $a = 2A$, $b = 2B$, $c = 2C$. Then

$$N_{8/4}(\eta)^2 = \varepsilon^{2a}\xi_1^{2b}\xi_2^{2c} = \varepsilon^{4A}\xi_1^{4B}\xi_2^{4C},$$

and hence

$$N_{8/4}\left(\frac{\eta}{\varepsilon^A\xi_1^B\xi_2^C}\right)^2 = 1 \quad \text{and} \quad N_{8/4}\left(\frac{\eta}{\varepsilon^A\xi_1^B\xi_2^C}\right) = \pm 1.$$

Therefore

$$\frac{\eta}{\varepsilon^A\xi_1^B\xi_2^C} \in U_0 = \langle -1, \eta_1, \eta_2, \eta_3, \eta_4 \rangle,$$

and hence $\eta \in \langle -1, \varepsilon, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4 \rangle = U'$.

Now let us look at the norm $N_{8/4}(\eta)$ for $\eta \in U_0'$. Since $N_{4/1}(N_{8/4}(\eta)) = N_{8/1}(\eta) = 1$, Proposition 7 tells us that

$$N_{8/4}(\eta) = \pm\varepsilon^a \xi_1^b \xi_2^c \text{ for some } a, b, c \in \mathbf{Z}.$$

The above result says that $\eta \in U'$, whenever a, b, c are all even. Therefore

$$U' = \{\eta \in U_0' | N_{8/4}(\eta) = \pm\varepsilon^{\text{even}} \xi_1^{\text{even}} \xi_2^{\text{even}}\}.$$

All other possibilities, $\{\eta \in U_0' | N_{8/4}(\eta) = \pm\varepsilon^{\text{odd}} \xi_1^{\text{even}} \xi_2^{\text{even}}\}, \ldots$, are cosets of U' in U_0'. Hence the index $[U_0' : U']$ divides 8, and the result follows.

Putting everything together, we have the following:

Theorem 4. *Let U be the unit group of K_n, and let U' be the subgroup generated by the units $-1, \varepsilon, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4$. Then we have the following:*

(a) *The index $q_1 = [U : U']$ is 2^r, $r \le 4$.*

(b) *If $N_{8/1}(\eta) = 1$ $\forall \eta \in U$, then the index $q_1 = [U : U'] = 2^r$, $r \le 3$.*

Remark. The result here cannot be improved for general cyclic octic fields. Consider the field

$$\mathbf{Q}(\zeta_{17})^+ = \mathbf{Q}\left(\cos\frac{2\pi}{17}\right),$$

which is a real cyclic octic field and has class number 1. Let σ_3 be the Galois action determined by $\zeta_{17} \mapsto \zeta_{17}^3$. Then the Galois group of $\mathbf{Q}(\zeta_{17})^+$ over $\mathbf{Q}$ is

$$\langle\sigma_3\rangle \simeq \mathbf{Z}/8\mathbf{Z} \simeq \{\sigma_3, \sigma_9, \sigma_{10}, \sigma_{13}, \sigma_5, \sigma_{15}, \sigma_{11}, 1\}.$$

Note that $\mathbf{Q}(\zeta_{17})^+$ has a unique quadratic subfield $\mathbf{Q}(\sqrt{17})$, whose fundamental unit is

$$\varepsilon = 4 + \sqrt{17} \sim 8.12310562562.$$

By Theorem 8.2 of Washington's book [13], the cyclotomic units

$$\left\{\psi_a = \sin\frac{\pi a}{17} / \sin\frac{\pi}{17} \middle| a = 2, 3, 4, 5, 6, 7, 8\right\}$$

form a system of fundamental units. Thus $U = \langle -1, \psi_a | a = 2, 3, \ldots, 8\rangle$. Note that we have

$$\psi_a = \zeta_{17}^{(1-a)/2} \frac{1 - \zeta_{17}^a}{1 - \zeta_{17}}$$

and hence

$$\psi_a^{\sigma_i} = \frac{\psi_{ai}}{\psi_i} \quad \text{for } a, i = 2, 3, \ldots, 16.$$

Also, we have

$$\psi_a = \psi_{17-a} \quad \text{for } a = 2, 3, \ldots, 16.$$

$$N_{8/4}(\psi_a) = \psi_a^{1+\sigma_{-4}}, \qquad N_{8/2}(\psi_a) = \psi_a^{1+\sigma_{-8}+\sigma_{-4}+\sigma_{-2}}.$$

Note that we have

$$\varepsilon = 4+\sqrt{17} = \psi_2^{-1}\psi_3\psi_4^{-1}\psi_5\psi_6\psi_7\psi_8^{-1}.$$

Let

$$\xi_1 = \psi_2\psi_8\psi_4^{-1}, \qquad \xi_2 = \psi_3\psi_5\psi_6^{-1}\psi_7^{-1},$$
$$\eta_1 = \psi_2\psi_8^{-1}, \qquad \eta_2 = \psi_3^{-1}\psi_5, \qquad \eta_3 = \psi_4, \qquad \eta_4 = \psi_6\psi_7^{-1}.$$

Then $\eta_1, \eta_2, \eta_3, \eta_4$ form a basis of U_0, and ξ_1 and ξ_2 form a basis for U_{y0} (see Shen [9]). Therefore

$$\begin{pmatrix} \log|\varepsilon| \\ \log|\xi_1| \\ \log|\xi_2| \\ \log|\eta_1| \\ \log|\eta_2| \\ \log|\eta_3| \\ \log|\eta_4| \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \log|\psi_2| \\ \log|\psi_3| \\ \log|\psi_4| \\ \log|\psi_5| \\ \log|\psi_6| \\ \log|\psi_7| \\ \log|\psi_8| \end{pmatrix}.$$

A calculation yields that the determinant equals 16, and hence $[U:U'] = 16$. This is the case $r=4$ in Theorem 4.

Next, we calculate the index q_1' as follows: Clearly, we have $q_1' = R_1/R'$. Equation (3.1) implies that

$$R_1 = 16R(\varepsilon)R(y_1, y_2)R(\rho_1, \rho_2, \rho_3, \rho_4).$$

Exactly the same calculation yields

$$R' = 16R(\varepsilon)R(\xi_1, \xi_2)R(\eta_1, \eta_2, \eta_3, \eta_4).$$

Let $q_2 = [\langle \xi_1, \xi_2\rangle : \langle y_1, y_2\rangle]$, and let $q_3 = [\langle \eta_1, \eta_2, \eta_3, \eta_4\rangle : \langle \rho_1, \rho_2, \rho_3, \rho_4\rangle]$. Then

$$q_2 = \frac{R(y_1, y_2)}{R(\xi_1, \xi_2)} \quad \text{and} \quad q_3 = \frac{R(\rho_1, \rho_2, \rho_3, \rho_4)}{R(\eta_1, \eta_2, \eta_3, \eta_4)}.$$

Therefore $q_1' = q_2q_3$ and $q = q_1q_2q_3$.

To estimate q_2 and q_3, we need lower bounds for $R(\xi_1, \xi_2)$ and $R(\eta_1, \eta_2, \eta_3, \eta_4)$. Our approach below is similar to Cusick's [2]. Let $\varepsilon_1 \in U_{y0}$, and let $\varepsilon_2 = \varepsilon_1^\sigma$. Then by the definition of U_{y0}, we have

$$\varepsilon_1^{\sigma^2} = \pm\frac{1}{\varepsilon_1} \quad \text{and} \quad \varepsilon_2^{\sigma^2} = \pm\frac{1}{\varepsilon_2}.$$

Consider the norm $N_{2/1}(d_{4/2}(\varepsilon_1))$, where $d_{4/2}$ is the discriminant from $\mathbf{Q}(y)$ to $\mathbf{Q}(\sqrt{2})$. A calculation yields

$$\left(\varepsilon_1 \pm \frac{1}{\varepsilon_1}\right)^2 \left(\varepsilon_2 \pm \frac{1}{\varepsilon_2}\right)^2,$$

or

$$\left(1 \pm \frac{1}{\varepsilon_1^2}\right)^2 \left(1 \pm \frac{1}{\varepsilon_2^2}\right)^2 \varepsilon_1^2\varepsilon_2^2.$$

Replacing ε_i by $1/\varepsilon_i$ or possibly using ε_2 in place of ε_1 and $\varepsilon^{-1} = \pm\varepsilon_2^\sigma$ in place of ε_2 if necessary, we may assume $|\varepsilon_i| \geq 1$ for $i = 1, 2$. Then the above expression is less than or equal to $2^4\varepsilon_1^2\varepsilon_2^2$. Since $N_{2/1}(d_{4/2}(\mathbf{Q}(y))) = 2^5b^2$, we have $2^5b^2 \leq 2^4\varepsilon_1^2\varepsilon_2^2$. Therefore we have the following inequality by Cauchy-Schwarz:

$$\begin{aligned}\log(a^2) < \log(a^2+1) = \log(2b^2) &\leq 2\sum_{i=1}^{2}\log|\varepsilon_i| \\ &\leq 2\sqrt{2}\sqrt{\log^2|\varepsilon_1| + \log^2|\varepsilon_2|}.\end{aligned}$$

Hence we have

$$\log^2|\varepsilon_1| + \log^2|\varepsilon_2| > \tfrac{1}{2}\log^2 a. \tag{4.3}$$

Since $\{\xi_1, \xi_2\}$ generates $U_{y0}/\{\pm 1\}$,

$$\varepsilon_1 = \pm\xi_1^{x_1}\xi_2^{x_2} \quad \text{for some } x_1, x_2 \in \mathbf{Z}.$$

Therefore

$$\log|\varepsilon_1| = x_1\log|\xi_1| + x_2\log|\xi_2|,$$

and hence

$$\log|\varepsilon_2| = x_1\log|\xi_2| - x_2\log|\xi_1|.$$

Define a quadratic form $Q(x_1, x_2)$ in x_1, x_2 by

$$Q(x_1, x_2) = \log^2|\varepsilon_1| + \log^2|\varepsilon_2| = \sum_{i,j=1}^{2} a_{ij}x_ix_j.$$

A simple calculation gives that

$$Q(x_1, x_2) = R(\xi_1, \xi_2)(x_1^2 + x_2^2).$$

Thus $Q(1, 0) = R(\xi_1, \xi_2)$, and the inequality (29) gives a lower bound for $R(\xi_1, \xi_2)$, as follows:

$$R(\xi_1, \xi_2) > \tfrac{1}{2}\log^2 a. \tag{4.4}$$

Similarly, we can do the same thing as above to get a lower bound for $R(\eta_1, \eta_2, \eta_3, \eta_4)$. However, the situation is a little bit complicated. Let $\varepsilon_1 \in U_0$, and let $\varepsilon_i = \varepsilon_1^{\sigma^{i-1}}$, $i = 1, 2, 3, 4$. Then by the definition of U_0, we have $\varepsilon_i^{\sigma^4} = \pm 1/\varepsilon_i$, $i = 1, 2, 3, 4$. Consider the norm $N_{4/1}(d_{8/4}\varepsilon_1)$. A calculation yields

$$\left(\varepsilon_1 \pm \frac{1}{\varepsilon_1}\right)^2\left(\varepsilon_2 \pm \frac{1}{\varepsilon_2}\right)^2\left(\varepsilon_3 \pm \frac{1}{\varepsilon_3}\right)^2\left(\varepsilon_4 \pm \frac{1}{\varepsilon_4}\right)^2,$$

or

$$\left(1 \pm \frac{1}{\varepsilon_1^2}\right)^2 \cdots \left(1 \pm \frac{1}{\varepsilon_4^2}\right)^2 \varepsilon_1^2\varepsilon_2^2\varepsilon_3^2\varepsilon_4^2.$$

Again, we may assume that $|\varepsilon_i| \geq 1$ for $i = 1, 2$. Then the above expression is less than or equal to $2^8\varepsilon_1^2\varepsilon_2^2\varepsilon_3^2\varepsilon_4^2$. Since $N_{4/1}(d_{8/4}K_n) = 2^9b^2$,

$$2^9b^2 \leq 2^8\varepsilon_1^2\varepsilon_2^2\varepsilon_3^2\varepsilon_4^2.$$

Therefore, we have

$$2\log a = \log(a^2) < \log(a^2+1) = \log(2b^2) \leq 2\sum_{i=1}^{4}\log|\varepsilon_i|.$$

By Cauchy-Schwarz, the above is less than or equal to

$$2\sqrt{4}\left(\sum_{i=1}^{4}\log^2|\varepsilon_i|\right)^{1/2}.$$

Thus we have

$$\log^2|\varepsilon_1| + \log^2|\varepsilon_2| + \log^2|\varepsilon_3| + \log^2|\varepsilon_4| > \tfrac{1}{4}\log^2 a. \tag{4.5}$$

Since $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ generates $U_0/\{\pm 1\}$,

$$\varepsilon_1 = \pm\eta_1^{x_1}\eta_2^{x_2}\eta_3^{x_3}\eta_4^{x_4} \quad \text{for some } x_i \in \mathbf{Z}.$$

Therefore we have

$$\log|\varepsilon_1| = x_1\log|\eta_1| + x_2\log|\eta_2| + x_3\log|\eta_3| + x_4\log|\eta_4|,$$

and hence

$$\begin{aligned}
\log|\varepsilon_2| &= x_1\log|\eta_2| + x_2\log|\eta_3| + x_3\log|\eta_4| - x_4\log|\eta_1|,\\
\log|\varepsilon_3| &= x_1\log|\eta_3| + x_2\log|\eta_4| - x_3\log|\eta_1| - x_4\log|\eta_2|,\\
\log|\varepsilon_4| &= x_1\log|\eta_4| - x_2\log|\eta_1| - x_3\log|\eta_2| - x_4\log|\eta_3|.
\end{aligned}$$

Define a quadratic form $Q(x_1, x_2, x_3, x_4)$ in x_1, x_2, x_3, x_4 by

$$Q(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4}\log^2|\varepsilon_i| = \sum_{i,j=1}^{4} a_{ij}x_ix_j. \tag{4.6}$$

A calculation shows that the determinant of the quadratic form Q is

$$\det(Q) = \det(a_{ij}) = R(\eta_1, \eta_2, \eta_3, \eta_4)^2. \tag{4.7}$$

In fact,

$$(a_{ij}) = \begin{pmatrix} A & B & 0 & -B\\ B & A & B & 0\\ 0 & B & A & B\\ -B & 0 & B & A \end{pmatrix},$$

where $A = \sum_{i=1}^4 \log^2|\eta_i|$ and $B = \log|\eta_1|\log|\eta_2| + \log|\eta_2|\log|\eta_3| + \log|\eta_3|\log|\eta_4| - \log|\eta_4|\log|\eta_1|$. An old result due to A. Korkine and G. Zolotareff [5] gives us that there exist integers x_1, x_2, x_3, x_4 not all zero, such that

$$Q(x_1, x_2, x_3, x_4) \leq (4\det(Q))^{1/4}.$$

Putting (4.5)–(4.7) together, we get a lower bound of $R(\eta_1, \eta_2, \eta_3, \eta_4)$:

$$R(\eta_1, \eta_2, \eta_3, \eta_4) > 2^{-5}\log^4 a. \tag{4.8}$$

Since $R'/R = q_1 = 1, 2, 4, 8$ or 16,

$$R \geq 2^{-4}R'.$$

Recall the formula

$$R' = 16\log\varepsilon R(\xi_1, \xi_2)R(\eta_1, \eta_2, \eta_3, \eta_4).$$

Then (4.4) and (4.8) give us a lower bound for R as follows:

Theorem 5. *Let R be the regulator of the field K_n. Then a lower bound for R is*

$$(2^{-6}\log\varepsilon)\log^6 a \quad (\approx 0.0137714623\log^6 a).$$

This bound is much better than Silverman's bound, since we have used more information about the field K_n. So we have an upper bound for the index $q = [U : U_1]$, where $U_1 = \langle -1, \varepsilon, y_1, y_2, \rho_1, \rho_2, \rho_3, \rho_4\rangle$.

Corollary. *An upper bound for the index $q = [U : U_1]$ is $2^{10}R_yR_\rho\log^{-6} a$, where $R_y = R(y_1, y_2)$ and $R_\rho = R(\rho_1, \rho_2, \rho_3, \rho_4)$.*

Remark. As $a \to \infty$, from (3.3) we have

$$2^{10}R_yR_\rho\log^{-6} a \to 2^{10}.$$

So the index $q = [U : U_1]$ is uniformly bounded.

To emphasize this result, we state it as follows:

Theorem 6. *Let U be the unit group of K_n, and let U_1 be the subgroup generated by the units $\rho_1, \rho_2, \rho_3, \rho_4, y_1, y_2$, and ε. Then the index $q = [U : U_1]$ is bounded, independent of a.*

To conclude this section, let us go back to estimate the class numbers of our real cyclic octic fields K_n, which was one of our original goals mentioned in the very beginning of this section. Since we have a sequence of cyclic octic number fields K_n with the property (by Theorem 3)

$$\frac{[K_n : \mathbf{Q}]}{\log|d(K_n)|} = \frac{8}{\log(2^{31}b_n^6)} \to 0 \quad \text{as } n \to \infty,$$

we may apply Brauer-Siegel's theorem (see Lang [6, p. 321]), which yields

$$\frac{\log(hR)}{\log\sqrt{d(K_n)}} \to 1.$$

But $R_1/R = q$ is uniformly bounded by Theorem 6, thus

$$C'\log^6 a < R < C\log^6 a \quad \text{for some } C, C'.$$

Therefore we have

$$\frac{\log h}{\frac{1}{2}\log(d(K_n))} \to 1,$$

and hence

$$\log h \sim \tfrac{1}{2}\log(d(K_n)) \sim 3\log(a_n).$$

This means that the class number h of the field K_n is very large, as $n \to \infty$. So we have the following corollary of Theorem 6.

Corollary. *Let h be the class number of the field K_n. Then $\log h \sim 3\log(a_n)$ as $n \to \infty$.*

5. Examples and conjectures

We are trying to make a conjecture on the index $q = [U : U_1]$ mentioned in Theorem 6 for the octic field K_n.

Since $a_n + b_n\sqrt{2} = (\sqrt{2}+1)^{2n+1}$, we have the following recursion formulas and a table for a_n and b_n:

$$a_{n+1} = 3a_n + 4b_n, \qquad b_{n+1} = 2a_n + 3b_n.$$

Table for a_n and b_n

n	a_n	b_n
0	1	1
1	7	5
2	41	29
3	239	169
4	1393	985
5	8119	5741
6	47321	33461
7	275807	195025
8	1607521	1136689
9	9369319	6625109
10	54608393	38613965
11	318281039	225058681
12	1855077841	1311738121
13	10812186007	7645370045
14	63018038201	44560482149
15	367296043199	259717522849

Note that since $b_3 = 13^2$ and $b_7 = 5^2 \times 7801$ are not squarefree, both are excluded from our discussion.

Now recall formula (27),

$$h = \frac{2^8 b^3 \log \varepsilon}{R} |L(1, \chi_4)|^2 |L(1, \chi_8)|^2 |L(1, \chi_8^3)|^2 .$$

To simplify the calculation of $|L(1, \chi)|$ we need the following:

Proposition 9. *Let χ be a primitive character with conductor f, which is divisible by 4. Then $\chi(a + \frac{f}{2}) = -\chi(a)$, and hence $\chi(\frac{f}{2} - a) = -\chi(a)$ if χ is even.*

Proof. Since $4|f$,

$$\left(1 + \frac{f}{2}\right)^2 = 1 + f + \frac{f}{4} f \equiv 1 \pmod{f}.$$

Therefore, $\chi(1 + \frac{f}{2}) = \pm 1$. For a odd, since $\frac{f}{2}(a - 1) \equiv 0 \pmod{f}$,

$$a\left(1 + \frac{f}{2}\right) = a + a\frac{f}{2} \equiv a + \frac{f}{2} \pmod{f},$$

and hence

$$\chi\left(a + \frac{f}{2}\right) = \chi(a)\chi\left(1 + \frac{f}{2}\right).$$

Note that $\chi(1 + \frac{f}{2})$ must be -1, otherwise $\chi(a + \frac{f}{2}) = \chi(a)$ $\forall \text{odd}\, a$; which means that χ is not primitive, a contradiction. Therefore $\chi(a + \frac{f}{2}) = -\chi(a)$ for all odd a. For even a, both are zero. This completes the proof.

Corollary. *Let χ be a primitive even character with conductor f which is divisible by 4. Then*

$$|L(1, \chi)| = \frac{2}{\sqrt{f}} \left| \sum_{a < f/4} \chi(a) \log \tan\left(\frac{a\pi}{f}\right) \right| . \tag{5.1}$$

Proof. If follows from (4.2) and the above proposition

Now we are ready to set up the calculations of the quotient q/h, for the first few examples, $n = 0, 1, 2$. Let us denote the quartic characters of K_n by χ_{16b}, $\overline{\chi}_{16b} = \chi_{16b}^3$, and denote the octic ones by χ_{32b}, $\chi'_{32b} = \chi_{32b}^3$, $\overline{\chi}'_{32b} = \chi_{32b}^5$, $\overline{\chi}_{32b} = \chi_{32b}^7$. With $q = R_1/R$ in mind, put (3.1), (4.1), and (5.1) together; we have the following formula for q/h:

Theorem 7. *Let h be the class number of K_n, and let $q = [U : U_1]$ be the index mentioned in Theorem 6. Then we have*

$$\frac{q}{h} = \frac{16 R_y R_\rho}{L_{16b} L_{32b} L'_{32b}},$$

where

$$R_y = \log^2|y_1| + \log^2|y_2|, \qquad y_i = \frac{1}{2}\left(\rho_i - \frac{1}{\rho_i}\right), \quad i = 1, 2;$$

$$R_\rho = \pm\det\begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & -A_1 \\ A_3 & A_4 & -A_1 & -A_2 \\ A_4 & -A_1 & -A_2 & -A_3 \end{vmatrix}, \qquad A_i = \log|\rho_i|, \; i = 1, 2, 3, 4;$$

$$L_{16b} = \left|\sum_{k<4b} \chi_{16b}(k)\log\tan\left(\frac{k\pi}{16b}\right)\right|^2,$$

$$L_{32b} = \left|\sum_{k<8b} \chi_{32b}(k)\log\tan\left(\frac{k\pi}{32b}\right)\right|^2,$$

$$L'_{32b} = \left|\sum_{k<8b} \chi^3_{32b}(k)\log\tan\left(\frac{k\pi}{32b}\right)\right|^2.$$

In the examples below, to determine which is the correct choice of χ, we have used Proposition 25 in Chapter 1 of Lang's book [6], along with the following fact:

Fact. Let p be an unramified rational prime in a number field K. Then we have

$$\chi(p) = \begin{cases} 1, & g = 8 \ (p \text{ splits}), \\ -1, & g = 4, \\ \pm i, & g = 2, \\ \pm\zeta_8^{1,3,5,7}, & g = 1 \ (p \text{ inert}), \end{cases}$$

where g is the number of primes in K lying above p, and χ is a generator of the character group.

Remarks. (a) Note also our χ's are always even characters, since our number fields are all totally real.

(b) We know that $(\mathbf{Z}/16\mathbf{Z})^\times$ is generated by 3 and -1, thus χ_{16} is determined by $\chi_{16}(3)$ and $\chi_{16}(-1)$. Clearly, $\chi_{16}(3) = \pm i$ and $\chi_{16}(-1) = \pm 1$.

(c) Similarly, we have $\chi_{32}(3) = \pm\zeta_8^{1,3,5,7}$ and $\chi_{32}(-1) = \pm 1$.

Here are the calculations: Let $\zeta = \zeta_8 = \exp(2\pi i/8)$.

(A) For $n = 0$, $a = 1$, $b = 1$: $f = 16$

$$\begin{aligned} &\rho_1 \sim 10.1531703876, && A_1 \sim 2.31778601017, \\ &\rho_2 \sim 1.87086841179, && A_2 \sim 0.626402714402, \\ &\rho_3 \sim -1.21850352559, && A_3 \sim 0.197623487464, \\ &\rho_4 \sim -3.29655820893, && A_4 \sim 1.19287895721, \\ &y_1 \sim 5.0273394921, && y_2 \sim 0.66817863792. \end{aligned}$$

Therefore,

$$R_\rho \sim 50.4046133952, \qquad R_y \sim 2.77044268466.$$

Note that the quartic character χ_{16} is determined by $\chi_{16}(3) = i$ and $\chi_{16}(-1) = 1$, and the octic character χ_{32} is determined by $\chi_{32}(3) = \zeta$ and $\chi_{32}(-1) = 1$. Therefore, we have

$$L_{16} = \left|\sum_{a<4} \chi_{16}(a) \log\tan\left(\frac{a\pi}{16}\right)\right|^2 \sim 2.77044268468,$$

$$L_{32} = \left|\sum_{a<8} \chi_{32}(a) \log\tan\left(\frac{a\pi}{32}\right)\right|^2 \sim 8.57489233007,$$

$$L'_{32} = \left|\sum_{a<8} \chi_{32}^3(a) \log\tan\left(\frac{a\pi}{32}\right)\right|^2 \sim 5.87816247538.$$

A calculation yields $q/h \sim 16.0000009745$. Therefore we must have $q/h = 16$ in the field K_0. Note that K_0 is contained in the cyclotomic field $\mathbf{Q}(\zeta_{32})$, whose class number is 1. Since $\mathbf{Q}(\zeta_{32})/K_)$ is totally ramified, we must have $h = 1$, by class field theory (Theorem 10.1 in Washington's book [13]), and hence $q = 16$.

Remarks. (a) Observe that we have $L_{16} = R_y$ and $L_{32}L'_{32} = R_\rho$.

(b) Let

$$B_1 = \log\tan(\pi/32), \qquad B_2 = \log\tan(5\pi/32),$$
$$B_3 = \log\tan(7\pi/32), \qquad B_4 = \log\tan(3\pi/32).$$

Then

$$L_{32}L'_{32} = \sum_{i=1}^{4} B_i^4 + 2(B_1^2B_3^2 + B_2^2B_4^2) + 4(B_1B_2 + B_3B_4)(B_1B_4 - B_2B_3),$$

which is the determinant as in R_ρ, i.e.,

$$L_{32}L'_{32} = \pm \begin{vmatrix} B_1 & B_2 & B_3 & B_4 \\ B_2 & B_3 & B_4 & -B_1 \\ B_3 & B_4 & -B_1 & -B_2 \\ B_4 & -B_1 & -B_2 & -B_3 \end{vmatrix}.$$

(c) Even better, we have

$$\rho_1 = \cot(\pi/32), \qquad \rho_2 = \cot(5\pi/32),$$
$$\rho_3 = -\cot(7\pi/32), \qquad \rho_4 = \cot(3\pi/32).$$

(B) For $n = 1$, $a = 7$, $b = 5$: $f = 160$

$$\rho_1 \sim 56.3729885432, \qquad A_1 \sim 4.03199011724,$$
$$\rho_2 \sim 2.29805856808, \qquad A_2 \sim 0.832064665647,$$
$$\rho_3 \sim -1.03611869347, \qquad A_3 \sim 0.0354817062613,$$
$$\rho_4 \sim -2.54076252734, \qquad A_4 \sim 0.932464243591,$$
$$y_1 \sim 28.1776247756, \qquad y_2 \sim 0.931454324485.$$

Therefore,

$$R_\rho \sim 317.319066816, \qquad R_y \sim 11.1508127982.$$

Note that the quartic character is $\chi_{80} = \chi_5\chi_{16}$, where χ_5 is quadratic and is uniquely determined by $\chi_5(2) = -1$. It is easy to see that χ_5 is even, so we must choose an even χ_{16}; one choice for χ_{16} is determined by $\chi_{16}(3) = i$ and $\chi_{16}(-1) = 1$. Therefore, we have

$$L_{80} = \left|\sum_{a<20} \chi_{80}(a) \log\tan\left(\frac{a\pi}{80}\right)\right|^2 \sim 22.3016255964.$$

For the octic character $\chi_{160} = \chi_5\chi_{32}$, where χ_5 is quartic and $\chi_5(2)$ has two choices, both are odd, and we choose the one determined by $\chi_5(2) = i$. Thus we need odd octic χ_{32}. There are four choices for $\chi_{32}(3)$. To make a right decision, we use Proposition 25 in Lang's book [6]. Reduced to modulo 23, $Q(X)$ is factored into the product of four squares of irreducibles; this says $g = 4$ and $\chi_{160}(23)$ must be -1, which forces us to choose $\chi_{32}(3) = \zeta$ and $\chi_{32}(-1) = -1$. Therefore, we have

$$L_{160} = \left|\sum_{a<40} \chi_{160}(a) \log\tan\left(\frac{a\pi}{160}\right)\right|^2 \sim 59.189069019$$

and

$$L'_{160} = \left|\sum_{a<40} \chi^3_{160}(a) \log\tan\left(\frac{a\pi}{160}\right)\right|^2 \sim 10.7222176717.$$

A calculation yields $q/h \sim 4.00000032779$, and hence we must have $q/h = 4$. Note also, $L_{80} = 2R_y$ and $L_{160}L'_{160} = 2R_\rho$. To calculate the class number h, we use Theorem 3 of Van der Linden's paper [12], and Theorem 3.5 in Washington's book [13] as follows:

Since K_1 and its Hilbert class field $H(K_1) = H$ are both in the maximal real subfield $\mathbf{Q}(\zeta_{160})^+$ of the cyclotomic field $\mathbf{Q}(\zeta_{160})$, by Van der Linden's results and

$$[\mathbf{Q}(\zeta_{160}) : \mathbf{Q}] = \phi(160) = 64, \qquad [K_1 : \mathbf{Q}] = 8,$$

we have $h = [H : K_1]$ divides 4. Therefore $h = 1, 2$ or 4. Thus we have

$$h = [H : K_1][\langle\chi_5, \chi_{32}\rangle_{\text{even}} : \langle\chi_{160}\rangle].$$

Since the order of χ_{160} is 8, and clearly

$$\langle\chi_5, \chi_{32}\rangle = \{\chi_5^a\chi_{32}^b | a + b = \text{even}\}$$

has order 16, we have $h = 2$, and hence $q = 8$ in this case.

(C) For $n = 2$, $a = 41$, $b = 29$: $f = 928$

$$\begin{aligned} \rho_1 &\sim 328.064014277, & A_1 &\sim 5.79320875482, \\ \rho_2 &\sim 2.39355130221, & A_2 &\sim 0.872778163592, \\ \rho_3 &\sim -1.00611501086, & A_3 &\sim 0.0060939005349, \\ \rho_4 &\sim -2.43518218298, & A_4 &\sim 0.890021572208, \\ y_1 &\sim 164.030483046, & y_2 &\sim 0.98788102681. \end{aligned}$$

Therefore,

$$R_\rho \sim 1233.06032039, \qquad R_y \sim 26.0106819565.$$

The quartic character is $\chi_{464} = \chi_{29}\chi_{16}$, where χ_{29} is quadratic and even, which is determined by $\chi_{29}(2) = -1$. Thus we must choose χ_{16} even, which is determined by $\chi_{16}(3) = i$ and $\chi_{16}(-1) = 1$. Therefore, we have

$$L_{464} = \left| \sum_{a<116} \chi_{464}(a) \log\tan\left(\frac{a\pi}{464}\right) \right|^2 = |A + iB|^2 \sim 52.021363913,$$

where

$$A = \log\left(\frac{\tan(\pi/464)\cdots\tan(111\pi/464)}{\tan(7\pi/464)\cdots\tan(113\pi/464)}\right) \sim \sqrt{26.1350518997}$$

and

$$B = \log\left(\frac{\tan(11\pi/464)\cdots\tan(115\pi/464)}{\tan(3\pi/464)\cdots\tan(107\pi/464)}\right) \sim \sqrt{25.8863120133}.$$

Finally the octic character is $\chi_{928} = \chi_{29}\chi_{32}$, where χ_{29}, which is determined by $\chi_{29}(2) = -i$, is odd, and hence χ_{32} must be odd. As before, a correct one is determined by $\chi_{32}(3) = \zeta$ and $\chi_{32}(-1) = -1$. Therefore, we have

$$L_{928} = \left| \sum_{a<232} \chi_{928}(a) \log\tan\left(\frac{a\pi}{928}\right) \right|^2 = |B_1 + \zeta B_2 + iB_3 + \zeta^3 B_4|^2,$$

where

$$\begin{aligned} B_1 &= \log\left(\frac{\tan(\pi/928)\cdots\tan(215\pi/928)}{\tan(33\pi/928)\cdots\tan(225\pi/928)}\right) \sim 0.849438364885, \\ B_2 &= \log\left(\frac{\tan(21\pi/928)\cdots\tan(229\pi/928)}{\tan(11\pi/928)\cdots\tan(213\pi/928)}\right) \sim 10.707542956, \\ B_3 &= \log\left(\frac{\tan(7\pi/928)\cdots\tan(199\pi/928)}{\tan(9\pi/928)\cdots\tan(231\pi/928)}\right) \sim -8.44603006283, \\ B_4 &= \log\left(\frac{\tan(5\pi/928)\cdots\tan(219\pi/928)}{\tan(3\pi/928)\cdots\tan(221\pi/928)}\right) \sim -4.91537996272. \end{aligned}$$

Therefore,

$$L_{928}L'_{928} = \pm\det\begin{vmatrix} B_1 & B_2 & B_3 & B_4 \\ B_2 & B_3 & B_4 & -B_1 \\ B_3 & B_4 & -B_1 & -B_2 \\ B_4 & -B_1 & -B_2 & -B_3 \end{vmatrix} \sim 41924.0508935.$$

Observe that

$$L_{928}L'_{928} = 34R_\rho\,, \qquad L_{464} = 2R_y\,.$$

So we have

$$\frac{q}{h} = \frac{16R_yR_\rho}{(2R_y)(34R_\rho)} = \frac{4}{17}\,.$$

To calculate the class number, we let L be the quartic field between $\mathbf{Q}(\zeta_{29})$ and $\mathbf{Q}$. Clearly, our field K_2 is contained in $L(\zeta_{29})$. Theorem 10.4(b) in Washington's book [13], implies $2 \nmid h(L)$. Since 2 is a primitive root modulo 29, we have 2 is inert in $\mathbf{Q}(\zeta_{29})/\mathbf{Q}$, and hence $(2) =$ prime in L. Thus, only one prime ramifies in $L(\zeta_{29})/L$. Theorem 10.4(a) in Washington's book implies $2 \nmid h(L(\zeta_{32}))$. Let $H_2 =$ 2-part of the Hilbert class field of K_2. Then

$$H_2 \subseteq L(\zeta_{32})\,, \quad \text{since } 2 \nmid h(L(\zeta_{32}))\,.$$

Again we have

$$[H_2 : K_2] = [\langle \chi_{29}\,, \chi_{32}\rangle_{\text{even}} : \langle \chi_{928}\rangle] = 2\,.$$

Therefore 2-part of $h(K_2)$ is 2, and probably $h = 2 \times 17 = 34$. Hence, probably, we have $q = 8$, and we know that q is 8 times an odd number.

(D) For $n = 4$, $a = 1393$, $b = 985$: $f = 31520$

$$R_y = 74.3994141935\,, \qquad R_\rho = 7812.96909763\,,$$

$$L_{16b}/R_y = 68\,, \qquad L_{32b}L'_{32b}/R_\rho = 4964 = 4 \cdot 1241\,.$$

Hence $h/q = 1241 \cdot 17$. As above, genus theory implies that $8|h$, but we do not know the exact power of 2 for h. This happens probably because 985 is a composite number. Therefore the only thing we can say is

$$8|q\,,$$

and hence probably we have $q = 8$.

(E) For $n = 5$, $a = 8119$, $b = 5741$: $f = 183712$

$$R_y = 107.915875995\,, \qquad R_\rho = 15463.2083598\,,$$

$$L_{16b}/R_y = 130 = 2 \cdot 65\,, \qquad L_{32b}L'_{32b}/R_\rho = 135458 = 2 \cdot 67729\,.$$

Hence $h/q = (67729 \cdot 65)/4$. As above, genus theory implies that $2|h$. Since 2 is a primitive root mod 5741, 2 is inert in the field $\mathbf{Q}(\zeta_{5741})$. This implies that 2 does not divide the class number of the genus field. Therefore $2\|h$, and hence $2^3\|q$. Again we have

$$q = 8 \cdot (\text{odd})\,,$$

and hence probably we have $q = 8$.

So now from our examples above, we suspect that for $n > 0$, the index $q = 8$. For $n = 0$, the index $q = 16$, which is probably because the conductor of K_0 is a prime power. Therefore we may make the following conjecture.

Conjecture 1. For $n > 0$ with b_n squarefree, the index $q = [U : U_1]$ is 8.

From this conjecture, we should be able to find some unit whose square root is also a unit in U. To do so, it is better to observe what happens to the quartic subfield $\mathbf{Q}(y)$. By exactly the same argument as in Theorem 7, we have the following:

Proposition 10. *Let h_y be the class number of $\mathbf{Q}(y)$, and let $q_y = [U_y : U_{1y}]$, where U_y is the full unit group of $\mathbf{Q}(y)$, and $U_{1y} = \langle \pm 1, \varepsilon, y_1, y_2 \rangle$. Then we have*

$$\frac{q_y}{h_y} = \frac{2R_y}{L_{16b}},$$

where $R_y = \log^2|y_1| + \log^2|y_2|$ and

$$L_{16b} = \left| \sum_{k<4b} \chi_{16b}(k) \log\tan\left(\frac{k\pi}{16b}\right) \right|^2 .$$

A calculation yields $q_y = 2$ for $n = 0, 1, 2$. This says that there exists a unit in U_{1y}, whose square root is a unit in U_y. It is clear that such a unit must be totally positive and it is easy to see that $y_1 y_2 \varepsilon$ is one such unit in U_{1y}. We know $y_1 = y$, $y_2 = (y-1)/(y+1)$, and

$$\begin{aligned} z &= \tfrac{1}{4}(\rho_1 + \rho_3 + \rho_5 + \rho_7) = \tfrac{1}{2}(y - \tfrac{1}{y}) \\ &= a + \sqrt{a^2+1} = a + \sqrt{2}b = \varepsilon^{2n+1}, \end{aligned}$$

by equation (9). So we have the following:

$$y_1 y_2 \varepsilon = y_1 y_2 z \varepsilon^{-2n} = y\left(\frac{y-1}{y+1}\right)\left(\frac{y^2-1}{2y}\right)\varepsilon^{-2n} = \frac{(y-1)^2}{2\varepsilon^{2n}}.$$

Therefore

$$\sqrt{y_1 y_2 \varepsilon} = \frac{y-1}{\sqrt{2}\varepsilon^n}, \quad \text{since } y > 1.$$

Let $\xi = (y-1)/(\sqrt{2}\varepsilon^n)$; then $\xi \in \mathbf{Q}(y)$ and ξ is a unit. Therefore $\xi \in U_y$. Now the units ε, y, ξ are independent, and the index $[U_y : \langle \pm 1, y, \xi \rangle]$ must be 1 for $n = 0, 1, 2$. Therefore they form a system of fundamental units of the quartic subfield $\mathbf{Q}(y)$ for $n = 0, 1, 2$. In general, M.-N. Gras [4] showed that $q_y = 1$ or 2, in any simplest quartic field. Thus the index q_y in our quartic subfield $\mathbf{Q}(y)$ must be 2 for all n with b_n squarefree, and hence we have the following:

Theorem 8. *In the quartic subfield $\mathbf{Q}(y)$ of the octic field K_n, let h_y be its class number, and let $q_y = [U_y : U_{1y}]$, where U_y is the full unit group of $\mathbf{Q}(y)$, and $U_{1y} = \langle \pm 1, \varepsilon, y_1, y_2 \rangle$. Then for all $n \in \mathbf{N}$ with b_n squarefree, we have*

(a) $q_y = 2$,

(b) *the units $\varepsilon, y, \xi = (y-1)/(\sqrt{2}\varepsilon^n)$ form a system of fundamental units for $\mathbf{Q}(y)$,*

(c)

$$h_y = \left| \sum_{k<4b} \chi_{16b}(k) \log\tan\left(\frac{k\pi}{16b}\right) \right|^2 \Big/ \left[\log^2 y + \log^2\left(\frac{y-1}{y+1}\right)\right],$$

(d) $\log h_y \sim \log a$, *as* $n \to \infty$.

Now let us go back to the octic field K_n. We are looking for some totally positive units in U_1. It is easy to see $-\rho_1\rho_3 y_1$ is totally positive and

$$-\rho_1\rho_3 y_1 = -\rho\left(\frac{-\rho-1}{\rho-1}\right)\left(\frac{\rho^2-1}{2\rho}\right) = \frac{(\rho+1)^2}{2}.$$

Therefore $\sqrt{-\rho_1\rho_3 y_1} = (\rho+1)/\sqrt{2}$. Let $\eta = (\rho+1)/\sqrt{2}$, and let $\eta_k = \eta^{\sigma_{k-1}}$, $k = 1, 2, 3, \dots, 8$. Then

$$\eta_k = \frac{\rho_k + 1}{(-1)^{k-1}\sqrt{2}}, \qquad k = 1, 2, 3, \dots, 8.$$

Therefore we have

$$\prod_{k=1}^{8} \eta_k = \frac{1}{16}\prod_{k=1}^{8}(\rho_k + 1) = \frac{1}{16}P(1) = 1,$$

by equation (3). For the sum, we have

$$\sum_{k=1}^{8} \eta_k = \frac{1}{\sqrt{2}}(4z_1 - 4z_2) = 2\sqrt{2}(z_1 - z_2).$$

But $z_1 + z_2 = 2a$ and $z_1 z_2 = -1$ imply

$$z_1 - z_2 = \sqrt{4a^2 + 4} = 2\sqrt{2}b.$$

Therefore

$$\sum_{k=1}^{8} \eta_k = 8b.$$

It is very easy to see that η is a unit in U, since η^2 is a unit and $\eta \in K_n$. However, to work out its minimal polynomial takes a lot of time. A calculation yields that the minimal polynomial of η is

$$X^8 - 8bX^7 - 4aX^6 + 40bX^5 - 6X^4 - 40bX^3 + 4aX^2 + 8bX + 1.$$

Clearly, the units $\varepsilon, y, \xi, \rho_1, \rho_2, \eta_1, \eta_2$ are independent in U, and for $n = 1, 2$, the index

$$[U : \langle \pm 1, y, \xi, \rho_1, \rho_2, \eta_1, \eta_2 \rangle] = 1.$$

Therefore these seven units form a system of fundamental units for $n = 1, 2$. Hence we make the following conjecture for all $n > 0$.

Conjecture 2. The units ε, y, ξ, ρ_1, ρ_2, η_1, η_2 in the ring of algebraic integers of the field K_n form a system of fundamental units for all $n > 0$, such that b_n is squarefree.

6. A possible generalization to 2^n-tic fields

Now, let us analyse what we have so far. In the simplest quadratic field, we have the polynomial

$$P_2(X) = X^2 - aX - 1 = (X^2 - 1) - aX.$$

Let $Q_1(X) = X$, and let $Q_2(X) = X^2 - 1$. Then

$$P_2(X) = Q_2(X) - aQ_1(X).$$

In the simplest quartic field, we have the polynomial

$$P_4(X) = X^4 - aX^3 - 6X^2 + aX + 1 = (X^4 - 6X^2 + 1) - aX(X^2 - 1).$$

Let $Q_4(X) = X^4 - 6X^2 + 1$. Then

$$P_4(X) = Q_4(X) - aQ_1(X)Q_2(X).$$

In the simplest octic field, we have the polynomial

$$\begin{aligned} P_8(X) &= X^8 - aX^7 - 28X^6 + 7aX^5 + 70X^4 - 7aX^3 - 28X^2 + aX + 1 \\ &= (X^8 - 28X^6 + 70X^4 - 28X^2 + 1) - aX(X^2 - 1)(X^4 - 6X^2 + 1). \end{aligned}$$

Let $Q_8(X) = X^8 - 28X^6 + 70X^4 - 28X^2 + 1$. Then

$$P_8(X) = Q_8(X) - aQ_1(X)Q_2(X)Q_4(X).$$

Note that

$$\begin{aligned} Q_1(X) = Q_{2^0}(X) = X &= \Re(X+i), \\ Q_2(X) = X^2 - 1 &= \Re(X+i)^2, \\ Q_4(X) = X^4 - 6X^2 + 1 = X^4 - \binom{4}{2}X^2 + 1 &= \Re(X+i)^4, \end{aligned}$$

and

$$Q_8(X) = X^8 - \binom{8}{2}X^6 + \binom{8}{4}X^4 - \binom{8}{6}X^2 + 1 = \Re(X+i)^8.$$

Hence the next one, $Q_{16}(X)$, should be the real part of $(X+i)^{16}$, and

$$P_{16}(X) = Q_{16}(X) - aQ_1(X)Q_2(X)Q_4(X)Q_8(X).$$

In general, for $n = 0, 1, 2, \dots$, let $Q_{2^n}(X)$ be the real part of $(X+i)^{2^n}$, which is

$$X^{2^n} - \binom{2^n}{2}X^{2^n-2} + \binom{2^n}{4}X^{2^n-4} - + \cdots + \binom{2^n}{2^n-4}X^4 - \binom{2^n}{2^n-2}X^2 + 1.$$

Let

$$P_{2^n}(X) = Q_{2^n}(X) - a\prod_{k=0}^{n-1} Q_{2^k}(X) \quad \text{for } a \in \mathbf{Z},\ n = 1, 2, 3, \ldots.$$

Write $P_{2^n}(X; a) = P_{2^n}(X)$. Then $Q_{2^n}(X) = P_{2^n}(X; 0) = \Re(X+i)^{2^n}$. Note also that in fact, we have

$$P_{2^n}(X) = \Re(X+i)^{2^n} - \frac{a}{2^n}\Im(X+i)^{2^n}. \tag{6.1}$$

To prove equation (6.1), it suffices to prove

$$\prod_{k=0}^{n-1} \Re(X+i)^{2^k} = \frac{1}{2^n}\Im(X+i)^{2^n}.$$

But this is true by induction on n.

As expected, we have the following:

Proposition 11. *Let* $N = 2^n$. *Then* $P_N(X) = 0$ *has* 2^n *distinct real roots.*

Proof. The proof is by induction on n. See Shen [9].

Remark. Claim. If ρ is a real root of $P_N(X) = 0$, then so is $(\varepsilon\rho - 1)/(\rho + \varepsilon)$.

To see this, we let

$$\alpha = \rho + i, \qquad \beta = \frac{\varepsilon\rho - 1}{\rho + \varepsilon} + i.$$

Then

$$P_N(\rho) = \Re(\alpha^N) - \frac{a}{N}\Im(\alpha^N) = 0. \tag{6.2}$$

A calculation shows

$$\beta = \frac{\alpha}{\rho + \varepsilon}(\varepsilon + i).$$

Note that $\rho + \varepsilon \in \mathbf{R}$ and

$$(\varepsilon + i)^N = -(\Im(\varepsilon + i)^{N/2})^2 \in \mathbf{R},$$

since $\Re(\varepsilon + i)^{N/2} = Q_{N/2}(\varepsilon) = 0$, and hence $(\varepsilon + i)^{N/2} = i\Im(\varepsilon + i)^{N/2}$ is purely imaginary. Therefore $\beta^N = c\alpha^N$ for some $c \in \mathbf{R}$. Hence we have

$$\begin{aligned} P_N\left(\frac{\varepsilon\rho - 1}{\rho + \varepsilon}\right) &= \Re(\beta^N) - \frac{a}{N}\Im(\beta^N), \\ &= c\Re(\alpha^N) - \frac{a}{N}c\Im(\alpha^N), \\ &= c(\Re(\alpha^N) - \frac{a}{N}\Im(\alpha^N)), \\ &= 0 \quad \text{(by equation (6.2))}. \end{aligned}$$

This proves the claim.

Proposition 12. *Let ρ be a root of $P_{2^n}(X)=0$, and let ε be the largest root of $Q_{2^{n-1}}(X)=0$. Then the transformation $\sigma(\rho)=(\varepsilon\rho-1)/(\rho+\varepsilon)$ has order 2^n. In other words, the 2×2 matrix*

$$A=\begin{pmatrix}\varepsilon & -1\\ 1 & \varepsilon\end{pmatrix},$$

is of order 2^n, in the group $\mathbf{PGL}_2(\mathbf{R})$.

(The proof is given below.)

Examples. (a) For $n=1$. The root of $Q_1(X)=0$ is 0, thus $\varepsilon=0$, and hence $A=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ is of order 2.

(b) For $n=2$. The roots of $Q_2(X)=0$ are ± 1, thus $\varepsilon=1$, and hence $A=\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ is of order 4.

(c) For $n=3$. The roots of $Q_4(X)=0$ are $\pm(\sqrt{2}+1)$, $\pm(\sqrt{2}-1)$, thus $\varepsilon=\sqrt{2}+1$, and hence $A=\left(\begin{smallmatrix}\varepsilon & -1\\ 1 & \varepsilon\end{smallmatrix}\right)$ is of order 8.

(d) For $n=4$. The roots of $Q_8(X)=0$ are

$$\pm\left(\sqrt{2}+1\pm\sqrt{4+2\sqrt{2}}\right),\qquad \pm\left(\sqrt{2}-1\pm\sqrt{4-2\sqrt{2}}\right).$$

Therefore

$$\varepsilon=\sqrt{2}+1+\sqrt{4+2\sqrt{2}}\sim 5.02733949213,$$

and hence $A=\left(\begin{smallmatrix}\varepsilon & -1\\ 1 & \varepsilon\end{smallmatrix}\right)$ is of order 16.

Proof of Proposition 12. We prove that A is of order 2^n. Since A has two distinct eigenvalues $\varepsilon+i$, and $\varepsilon-i$, A must be similar to the diagonal matrix

$$D=\begin{pmatrix}\varepsilon+i & 0\\ 0 & \varepsilon-i\end{pmatrix}.$$

Clearly, A and D have the same order. Hence it suffices to show that D is of order 2^n. Now

$$D\rho=\frac{\varepsilon+i}{\varepsilon-i}\rho=\zeta\rho,$$

where $\zeta=(\varepsilon+i)/(\varepsilon-i)$. Note that ε is real and

$$Q_{2^{n-1}}(\varepsilon)=0=\Re(\varepsilon+i)^{2^{n-1}}.$$

Therefore

$$(\varepsilon+i)^{2^{n-1}}=\Re(\varepsilon+i)^{2^{n-1}}+i\Im(\varepsilon+i)^{2^{n-1}}=i\Im(\varepsilon+i)^{2^{n-1}},$$

and hence

$$(\varepsilon-i)^{2^{n-1}}=\overline{(\varepsilon+i)}^{2^{n-1}}=-i\Im(\varepsilon+i)^{2^{n-1}}.$$

All these imply

$$\zeta^{2^{n-1}}=\left(\frac{\varepsilon+i}{\varepsilon-i}\right)^{2^{n-1}}=-1,$$

and thus ζ is of order 2^n. But $D\rho=\zeta\rho$, so the result follows immediately.

Everything described above is very nice. However, when n is getting large, a lot of difficulties arise. In the octic case, we were able to select a family of a, so that the generating fields are cyclic, and $\mathbf{Q}(\sqrt{2}) = \mathbf{Q}(\varepsilon)$ is always the unique quadratic subfield in every field of the family. For $n \geq 4$, we have the same problem and the selection of a to make the generating fields cyclic or even Galois becomes much harder. Anyway, this is a possible generalization to 2^n-tic algebraic number fields, and still a lot of work remains to be done. So this is not the end of the story, instead it is the beginning of another story.

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